Discrete Mathematics 124 (1994) 37–47
North-Holland

A closure concept based on neighborhood unions of independent triples

H.J. Broersma

Department of Applied Mathematics, University of Twente, P.O. Box 217, 7500 AE Enschede, Netherlands

I. Schiermeyer

Technische Hochschule Aachen, Templergraben 55, W-52056 Aachen, Germany

Received 18 December 1990
Revised 19 September 1991

Abstract

The well-known closure concept of Bondy and Chvátal is based on degree-sums of pairs of nonadjacent (independent) vertices. We show that a more general concept due to Ainouche and Christofides can be restated in terms of degree-sums of independent triples. We introduce a closure concept which is based on neighborhood unions of independent triples and which also generalizes the closure concept of Bondy and Chvátal. Let G be a 2-connected graph on n vertices and let u, v be a pair of nonadjacent vertices of G. Define $\lambda_{uv}=|N(u)\cap N(v)|$, $T_{uv}=\{w\in V(G)-\{u,v\}\mid u,v\notin N(w)\}$ and $t_{uv}=|T_{uv}|$. We prove the following main result: If $\lambda_{uv}\geqslant 3$ and $|N(u)\cup N(v)\cup N(w)|\geqslant n-\lambda_{uv}$ for at least $t+2-\lambda_{uv}$ vertices $w\in T$, or if $\lambda_{uv}\leqslant 2$ and G satisfies the 1-2-3-condition (defined in Section 2) and $|N(u)\cup N(v)\cup N(w)|=n-3$ for all vertices $w\in T$, then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

1. Introduction

We use Bondy and Murty [4] for terminology and notation not defined here and consider simple graphs only.

Let G be a graph. If G has a Hamilton cycle (a cycle containing every vertex of G), then G is called *Hamiltonian*. The set of vertices adjacent to a vertex v of G is denoted by $N(v)$ and $d(v)=|N(v)|$. For a pair $\{u,v\}$ of nonadjacent vertices of G, we define $\lambda_{uv}=|N(u)\cap N(v)|$, $T_{uv}=\{w\in V(G)-\{u,v\}\mid u,\ v\notin N(w)\}$ and $t_{uv}=|T_{uv}|$. If u and v are clearly understood, we sometimes write λ instead of λ_{uv}, T instead of T_{uv} and t instead

Correspondence to: I. Schiermeyer, Technische Hochschule Aachen, Templergraben 55, W-52056 Aachen, Germany.

SSDI 0012-365X(92)00049-1

of t_{uv}. For a triple $\{u, v, w\}$ of mutually nonadjacent vertices of G, we define $\lambda_{uvw} = |N(u) \cap N(v) \cap N(w)|$.

The closure concept of Bondy and Chvátal [3] is based on the following result of Ore [8].

Theorem 1.1 (Bondy and Chvátal [3] and Ore [8]). *Let u and v be two nonadjacent vertices of a graph G of order n such that $d(u)+d(v) \geqslant n$. Then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.*

By successively joining pairs of nonadjacent vertices having degree-sum at least n as long as this is possible (in the new graph(s)), the unique so-called n-closure $C_n(G)$ is obtained. Using Theorem 1.1 it is easy to prove the following result.

Theorem 1.2 (Bondy and Chvátal [3]). *Let G be a graph of order n. Then G is Hamiltonian if and only if $C_n(G)$ is Hamiltonian.*

Corollary 1.3 (Bondy and Chvátal [3]). *Let G be a graph of order $n \geqslant 3$. If $C_n(G)$ is complete ($C_n(G) = K_n$), then G is Hamiltonian.*

It is well known that Corollary 1.3 generalizes a number of earlier sufficient degree conditions for Hamiltonicity (cf. [2, 5]). Ainouche and Christofides [1] established the following generalization of Theorem 1.1.

Theorem 1.4 (Ainouche and Christofides [1]). *Let u and v be two nonadjacent vertices of a 2-connected graph G and let $d_1 \leqslant d_2 \leqslant \cdots \leqslant d_t$ be the degree sequence of the vertices of T (in G). If*

$$d_i \geqslant t+2 \quad \textit{for all } i \textit{ with} \quad \max(1, \lambda-1) \leqslant i \leqslant t, \tag{1}$$

then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

In [1], the corresponding (unique) closure of G is called the 0-dual closure $C_0^*(G)$. Since Theorem 1.4 is more general than Theorem 1.1 (cf. [1]), $G \subseteq C_n(G) \subseteq C_0^*(G)$ (Here $\subseteq$ means " is a spanning subgraph of").

The counterpart of Corollary 1.3 is Corollary 1.5.

Corollary 1.5 (Ainouche and Christofides [1]). *Let G be a 2-connected graph. If $C_0^*(G)$ is complete, then G is Hamiltonian.*

Our first observation is that (1) can be restated in terms of degree-sums of independent triples.

Proposition 1.6. *Relation* (1) *is equivalent to*

$$d(u)+d(v)+d(w)\geq n+\lambda_{uv} \text{ for at least } \min(t,t+2-\lambda_{uv}) \text{ vertices } w\in T \text{ (where } n=|V(G)|). \tag{2}$$

Proof. Relation (1) can be restated as follows: $d(w)\geq t+2$ for at least $\min(t, t+2-\lambda_{uv})$ vertices $w\in T$. Substituting $t=n-2-d(u)-d(v)+\lambda_{uv}$ we obtain (2). □

Motivated by the above observation and the following recent result of Flandrin et al. [7], we were led to investigate closure concepts based on triples instead of pairs of nonadjacent vertices.

Theorem 1.7 (Flandrin et al. [7]). *Let G be a 2-connected graph of order n. If $d(u)+d(v)+d(w)\geq n+\lambda_{uvw}$ for all independent triples $\{u,v,w\}$ of vertices of G, then G is Hamiltonian.*

First, we tried to establish a result which would be more general than Theorem 1.4 by replacing $n+\lambda_{uv}$ in condition (2) by $n+\lambda_{uvw}$. However, the following examples show that this is impossible.

Let p,q,r be three natural numbers such that $p,q,r\geq 3$ and $p+q+r=n$. Let G_{pqr} denote the graph of Fig. 1(a) on n vertices obtained from three disjoint complete graphs $H_1=K_p$, $H_2=K_q$ and $H_3=K_r$ by adding the edges of two triangles between two disjoint triples of vertices, each containing one vertex of each of H_1, H_2 and H_3. Moreover, let G^+_{pqr} denote the graph of Fig. 1(b) obtained from G_{pqr} by adding an edge joining a vertex of H_1 and one of H_2, both not incident with edges of the added triangles.

It is easy to check that G_{pqr} is non-Hamiltonian, and that the addition of any new edge to G_{pqr} yields a Hamiltonian graph. In particular, G^+_{pqr} is Hamiltonian and $G_{pqr}+uv$ is Hamiltonian, where u and v are nonadjacent vertices of H_1 and H_2 (in G_{pqr}) which are both incident with edges of the added triangles. For these u and v, $d(u)+d(v)+d(w)=n+1>n+\lambda_{uvw}=n$ for all $w\in T$, while $G_{pqr}+uv$ is Hamiltonian and G_{pqr} is not. So we cannot replace $n+\lambda_{uv}$ in (2) by $n+\lambda_{uvw}$ in order to obtain a more

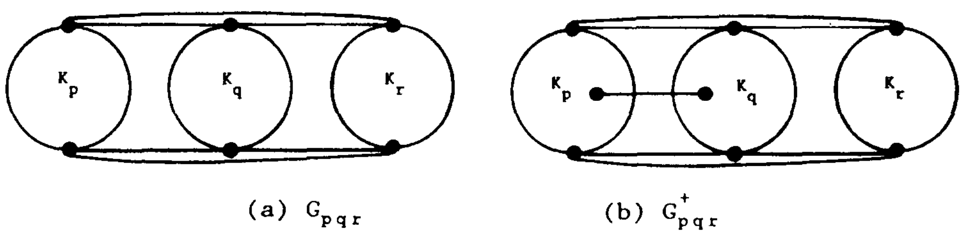

Fig. 1. G_{pqr} and G^+_{pqr}.

general result than Theorem 1.4. Moreover, there exist examples showing that replacing $n+\lambda_{uv}$ in (2) by $n+\lambda_{uvw}+c$, where c is a constant, is not enough to establish an analogue of Theorem 1.4.

However, by introducing a new condition and considering cardinalities of neighborhood unions instead of degree-sums, we were able to find another closure concept based on independent triples of vertices.

2. Results

Let u and v be two nonadjacent vertices of a 2-connected graph G of order n. Recall that $T=T_{uv}=\{w\in V(G)-\{u,v\}\mid u,v\notin N(w)\}$ and $t=|T|$. For a vertex $w\in T$, we let $\eta(w)=|N(w)-T|$, and we let $\eta_1\geqslant\eta_2\geqslant\cdots\geqslant\eta_t$ denote the ordered sequence corresponding to the set $\{\eta(w)\mid w\in T\}$. We say that G satisfies the 1-2-3-condition if $T=\emptyset$ or $\eta_i\geqslant 4-i$ for all i with $1\leqslant i\leqslant t$ (Note that $t\geqslant 1$ implies $\eta_1\geqslant 3$, $t\geqslant 2$ implies $\eta_2\geqslant 2$, and $t\geqslant 3$ implies $\eta_3\geqslant 1$).

In the next section we give a proof of the following result.

Theorem 2.1. *Let u and v be two nonadjacent vertices of a 2-connected graph G of order n.*

If $\lambda_{uv}\geqslant 3$ and

$$|N(u)\cup N(v)\cup N(w)|\geqslant n-\lambda_{uv} \quad \text{for at least } t+2-\lambda_{uv} \text{ vertices } w\in T, \tag{3}$$

or if $\lambda_{uv}\leqslant 2$ and G satisfies the 1-2-3-condition and

$$|N(u)\cup N(v)\cup N(w)|=n-3 \quad \text{for all vertices } w\in T, \tag{4}$$

then G is Hamiltonian if and only if $G+uv$ is Hamiltonian.

It is not difficult to see that we obtain a unique graph from G by successively joining pairs of nonadjacent vertices u and v satisfying the conditions of Theorem 2.1 as long as this is possible (in the new graph(s)). We call this graph the triple closure of G and denote it by $TC(G)$.

Proposition 2.2. *$C_n(G)\subseteq TC(G)$ for any graph G.*

Proof. Let u and v be two nonadjacent vertices of G with $d(u)+d(v)\geqslant n$. Since $t=n-2-d(u)-d(v)+\lambda$, this implies $\lambda\geqslant t+2$. If $\lambda=2$, then $t=0$, hence $T=\emptyset$, and clearly G satisfies the conditions of Theorem 2.1. If $\lambda\geqslant 3$, then $t+2-\lambda\leqslant 0$ implies that (3) is required for no vertices of T. Again G satisfies the conditions of Theorem 2.1. $\square$

In [6] Faudree et al. defined the $(n-2)$-neighborhood closure of a graph G, denoted by $N_{n-2}(G)$, as the (unique) graph obtained from G by successively joining pairs of nonadjacent vertices u and v satisfying $|N(u)\cup N(v)|\geqslant n-2$. Since for such pairs $T_{uv}=\emptyset$, it is clear that the following holds.

Proposition 2.3. $N_{n-2}(G)\subseteq TC(G)$ *for any graph* G.

Without proof we note that the graphs G^+_{pqr} have a complete triple closure, i.e., $TC(G^+_{pqr})=K_{p+q+r}$, while, if $p, q\geqslant 4$, $C^*_0(G^+_{pqr})=G^+_{pqr}$, $N_{n-2}(G^+_{pqr})=G^+_{pqr}$ and G^+_{pqr} does not satisfy the conditions of Theorem 1.7.

The graphs G_{pqr} show that we cannot omit the 1-2-3-condition in Theorem 2.1.

3. Proof of Theorem 2.1

We first introduce some additional terminology and notation.

For a Hamilton path $u=v_1v_2\cdots v_n=v$ from u to v we define $i^*=\max\{i\,|\,v_i\in N(u)\}$, $j^*=\min\{j\,|\,v_j\in N(v)\}$, where $i, j\in\{1,2,\ldots,n\}$. If $i^*>j^*$, then a constrained cycle is a cycle of the form $v_1v_2\cdots v_rv_nv_{n-1}\cdots v_sv_1$, where r and s $(s>r)$ are chosen in such a way that all vertices v_i with $r<i<s$, if any, belong to T_{uv}.

If P is a path of a graph G, we denote by $\vec{P}$ that path P with a given orientation; if x, $y\in V(P)$, then $x\vec{P}y$ denotes the consecutive vertices of P from x to y in the direction specified by $\vec{P}$. The same vertices, in reverse order, are given by $y\overleftarrow{P}x$. Analogous notation is used with respect to cycles instead of paths. Before proving Theorem 2.1 we establish two lemmas.

Lemma 3.1. *Let* $\vec{P}:u=v_1v_2\cdots v_n=v$ *be a Hamilton path of a 2-connected graph* G *with* $i^*>j^*$. *For a given constrained cycle* C_{uv}, *let* $X=\{v_i\,|\,v_i\notin V(C_{uv})\}$. *If* $\lambda_{uv}\geqslant 3$ *and*

$$|N(u)\cup N(v)\cup N(w)|\geqslant n-\lambda_{uv}\quad \text{for all vertices } w\in X \tag{5}$$

or if $\lambda_{uv}\leqslant 2$ *and* G *satisfies the* 1-2-3-*condition and*

$$|N(u)\cup N(v)\cup N(w)|\geqslant n-3 \text{ for all vertices } w\in X, \tag{6}$$

then G *is Hamiltonian.*

Proof. Assume G is not Hamiltonian and $C_{uv}=v_1v_2\cdots v_rv_nv_{n-1}\cdots v_sv_1$, where $2\leqslant r<s\leqslant n-1$. Clearly $X\neq\emptyset$; otherwise $v_1\vec{P}v_rv_n\overleftarrow{P}v_sv_1$ would be a Hamilton cycle.

If $\lambda_{uv}\geqslant 3$ there are $m\geqslant\lambda_{uv}-1$ constrained cycles $C_1,\ldots,C_m$ in G which induce pairwise disjoint subsets $X_1,\ldots,X_m$ of $V(G)$ with $X_i=V(G)-V(C_i)\neq\emptyset$ $(i=1,\ldots,m)$. Furthermore, $C_{uv}=C_k$ for some $k\in\{1,\ldots,m\}$. Assume $C_1,\ldots,C_m$ are ordered in such a way that the vertices of X_i are before the vertices of X_{i+1} on $\vec{P}$ $(i=1,\ldots,m-1)$. Let $C_i=v_1v_2\cdots v_{r(i)}v_nv_{n-1}\cdots v_{s(i)}v_1$ $(i=1,2,\ldots,m)$. If $k=1$, then by (5) there exists an integer $i\in\{2,\ldots,m\}$ such that $v_{s(1)-1}w\in E(G)$ for all vertices $w\in X_i$. Then $v_1\vec{P}v_{s(1)-1}v_{r(i)+1}\vec{P}v_nv_{r(i)}\overleftarrow{P}v_{s(1)}v_1$ is a Hamilton cycle, a contradiction. Hence $k\neq 1$. By similar arguments $k \neq m$. Now suppose $2 \leqslant k \leqslant m-1$. By (5) there exists an integer $i \in \{1, \ldots, k-1\}$ such that $wv_{r(k)+1} \in E(G)$ for all $w \in X_i$ or there exists an integer $j \in \{k+1, \ldots, m\}$ such that $v_{s(k)-1}w \in E(G)$ for all $w \in X_j$. Then $v_1 \vec{P} v_{s(i)-1} v_{r(k)+1} \vec{P} v_n v_{r(k)} \overleftarrow{P} v_{s(i)} v_1$ or $v_1 \vec{P}_{s(k)-1} v_{r(j)+1} \vec{P} v_n v_{r(j)} \overleftarrow{P} v_{s(k)} v_1$ is a Hamilton cycle, a contradiction.

Hence $\lambda_{uv} \leqslant 2$ and we may assume there is precisely one constrained cycle C_{uv}.

If, for some integer $i \in \{2, \ldots, r-1\}$, $v_i v_{r+1}, v_1 v_{i+1} \in E(G)$ or $v_i v_{s-1}, v_{i+1} v_n \in E(G)$, then $v_1 \vec{P} v_i v_{r+1} \vec{P} v_n v_r \overleftarrow{P} v_{i+1} v_1$ or $v_1 \vec{P} v_i v_{s-1} \overleftarrow{P} v_{i+1} v_n \overleftarrow{P} v_s v_1$ (respectively) is a Hamilton cycle, a contradiction.

If, for some integer $j \in \{s, \ldots, n-2\}$, $v_{r+1} v_{j+1}$, $v_1 v_j \in E(G)$ or $v_{s-1} v_{j+1}$, $v_j v_n \in E(G)$, then $v_1 \vec{P} v_r v_n \overleftarrow{P} v_{j+1} v_{r+1} \vec{P} v_j v_1$ or $v_1 \vec{P} v_{s-1} v_{j+1} \vec{P} v_n v_j \overleftarrow{P} v_s v_1$ (respectively) is a Hamilton cycle, a contradiction. Therefore, by (6) we get $X = T$ and

$$G[X] \text{ is complete.} \tag{7}$$

Let

$$p+1 = \min_{r+1 \leqslant i \leqslant s-1} \{i \mid \text{There is no } j \in \{2, \ldots, r-1\} \text{ with } v_j v_i,\ v_{j+1} v_n \in E(G) \text{ and there is no } j \in \{s, \ldots, n-2\} \text{ with } v_j v_n,\ v_i v_{j+1} \in E(G)\}.$$

By the above observations, $p+1$ is well defined.

Let

$$q-1 = \max_{p+1 \leqslant i \leqslant s-1} \{i \mid \text{There is no } j \in \{s, \ldots, n-2\} \text{ with } v_1 v_j,\ v_i v_{j+1} \in E(G) \text{ and there is no } j \in \{2, \ldots, r-1\} \text{ with } v_j v_i,\ v_1 v_{j+1} \in E(G)\}.$$

Then $q-1$ is well defined; otherwise the following Hamilton cycles contradict the assumptions.

If $p = r$:

$$v_1 \vec{P} v_i v_{r+1} \vec{P} v_n v_r \overleftarrow{P} v_{i+1} v_1 \quad \text{for some } i \in \{2, \ldots, r-1\}$$

or

$$v_1 \vec{P} v_r v_n \overleftarrow{P} v_{i+1} v_{r+1} \vec{P} v_i v_1 \quad \text{for some } i \in \{s, \ldots, n-2\}.$$

If $p > r$:

$$v_1 \vec{P} v_i v_{p+1} \vec{P} v_n v_{j+1} \vec{P} v_p v_j \overleftarrow{P} v_{i+1} v_1 \quad \text{for some } i, j \text{ with } 2 \leqslant i < j \leqslant r-1$$

or

$$v_1 \vec{P} v_j v_p \overleftarrow{P} v_{j+1} v_n \overleftarrow{P} v_{i+1} v_{p+1} \vec{P} v_i v_1 \quad \text{for some } i \in \{s, \ldots, n-2\} \text{ and } j \in \{2, \ldots, r-1\}$$

or

$$v_1 \vec{P} v_i v_{p+1} \vec{P} v_j v_n \overleftarrow{P} v_{j+1} v_p \overleftarrow{P} v_{i+1} v_1 \quad \text{for some } i \in \{2, \ldots, r-1\} \text{ and } j \in \{s, \ldots, n-2\}$$

or

$$v_1 \vec{P} v_p v_{j+1} \vec{P} v_n v_j \overleftarrow{P} v_{i+1} v_{p+1} \vec{P} v_i v_1 \quad \text{for some } i, j \text{ with } s \leqslant i < j \leqslant n-2.$$

Thus $X' = \{v_i \mid p+1 \leqslant i \leqslant q-1\} \neq \emptyset$ and, by the definition of $p+1$ and $q-1$,

$$N(w) \subseteq X \cup \{v_r, v_s\} \quad \text{for all } w \in X'.$$

If $p \geqslant r+2$, then by (7) $v_{p-1}v_{s-1} \in E(G)$ and $Q = v_p\vec{P}v_{s-1}v_{p-1}\overleftarrow{P}v_r$ is a path from v_p to v_r containing all vertices of $v_r\vec{P}v_{s-1}$.Then

$$v_1\vec{P}v_jv_p\vec{Q}v_r\overleftarrow{P}v_{j+1}v_n\overleftarrow{P}v_sv_1 \quad \text{for some } j \in \{2, \ldots, r-1\}$$

or

$$v_1\vec{P}v_r\overleftarrow{Q}v_pv_{j+1}\vec{P}v_nv_j\overleftarrow{P}v_sv_1 \quad \text{for some } j \in \{s, \ldots, n-2\}$$

is a Hamilton cycle, a contradiction. A similar contradiction is obtained if $p = r+1$ and $v_rv_i \in E(G)$ for some $i \in \{r+2, \ldots, s-1\}$, or if $q \leqslant s-2$, or if $q = s-1$ and $v_iv_s \in E(G)$ for some $i \in \{r+1, \ldots, s-2\}$.

Hence, we have $r \leqslant p \leqslant r+1$, $s-1 \leqslant q \leqslant s$. Furthermore, if $p = r+1$, $q = s-1$, then $t \geqslant 3$, $|X'| = t-2$ and $|N(w)| \leqslant t-1$ for all $w \in X'$; if $p = r+1, q = s$ or $p = r, q = s-1$, then $t \geqslant 2$, $|X'| = t-1$ and $|N(w)| \leqslant t$ for all $w \in X'$; if $p = r$, $q = s$, then $t \geqslant 1$, $|X'| = t$ and $|N(w)| \leqslant t+1$ for all $w \in X'$. In all cases, this contradicts the 1-2-3-condition. □

Lemma 3.2. *Let $\vec{P}: u = v_1v_2 \cdots v_n = v$ be a Hamilton path of a 2-connected graph G with $i^* \leqslant j^*$ satisfying the* 1-2-3-*condition. If*

$$|N(u) \cup N(v) \cup N(w)| = n-3 \quad \textit{for all vertices } w \in T, \tag{8}$$

then G is Hamiltonian.

Proof. Suppose G is not hamiltonian. By (8)

$$G[T] \text{ is complete.} \tag{9}$$

Let $A = \{v_i \mid i < i^*\}$, $B := \{v_j \mid j > j^*\}$, $D = \{v_i \mid i^* \leqslant i \leqslant j^*\}$ and distinguish the following three cases.

Case 1. $|D| = 1$. Clearly, $|D| = 1$ implies $i^* = j^*$ and, since G is 2-connected, there exists at least one edge v_pv_q in G with $v_p \in A$ and $v_q \in B$. Let $r = \min\{j > p \mid v_j \in N(v_1)\}$ and $s = \max\{j < p \mid v_j \in N(v_n)\}$. Among all possible edges v_pv_q, choose one for which $(r-p)+(q-s)$ is as small as possible. If $r = p+1$ and $s = q-1$, then $v_1\vec{P}v_pv_q\vec{P}v_nv_{q-1}\overleftarrow{P}v_{p+1}v_1$ is a Hamilton cycle, a contradiction.

Hence, we may assume $r > p+1$ and $s = q-1$; otherwise $v_{s+1} \in T$ and $v_{r-1}v_{s+1} \in E(G)$ by (9), contradicting the minimality of $(r-p)+(q-s)$. By the same argument we conclude that $T \cap B = \emptyset$.

If there exists an integer $i \in \{2, \ldots, p-1\}$ such that v_iv_{p+1}, $v_1v_{i+1} \in E(G)$ or an integer $j \in \{p+2, \ldots, i^*-1\}$ such that $v_{p+1}v_{j+1}$, $v_1v_j \in E(G)$, then

$$v_1\vec{P}v_iv_{p+1}\vec{P}v_{q-1}v_n\overleftarrow{P}v_qv_p\overleftarrow{P}v_{i+1}v_1 \quad \text{or} \quad v_1\vec{P}v_pv_q\vec{P}v_nv_{q-1}\overleftarrow{P}v_{j+1}v_{p+1}\vec{P}v_jv_1$$

is a Hamilton cycle, a contradiction.

Furthermore, if $v_{p-1}v_{r-1} \in E(G)$, then

$$v_1\vec{P}v_{p-1}v_{r-1}\overleftarrow{P}v_pv_q\vec{P}v_nv_{q-1}\overleftarrow{P}v_rv_1$$

is a Hamilton cycle, a contradiction.

Hence, $T=\{v_p, v_{p+1}, \ldots, v_{r-1}\}$ or $T=\{v_{p+1}, v_{p+2}, \ldots, v_{r-1}\}$.

If $T=\{v_p, v_{p+1}, \ldots, v_{r-1}\}$ then $t \geqslant 2$ and $|N(w)| \geqslant t+1$ for some vertex $w \in T-\{v_p\}$ since G satisfies the 1-2-3-condition. Let $w=v_j$ for some $j \in \{p+1, \ldots, r-1\}$. Then there exists (a) an integer $i \in \{r, \ldots, i^*-1\}$ such that $v_j v_{i+1} \in E(G)$ or (b) an integer $k \in \{2, \ldots, p-1\}$ such that $v_k v_j \in E(G)$. Choose j as small as possible among all $v_j \in \{v_{p+1}, \ldots, v_{r-1}\}$ with this property. If $j \leqslant r-2$, then there is a path Q_1 from v_j to v_i containing all vertices of $v_{p+1}\vec{P}v_i$ or a path Q_2 from v_j to v_r containing all vertices of $v_{p+1}\vec{P}v_r$ (by (9)). Then

$$v_1\vec{P}v_pv_q\vec{P}v_nv_{q-1}\overleftarrow{P}v_{i+1}v_j\vec{Q}_1v_iv_1 \quad \text{or} \quad v_1\vec{P}v_kv_j\vec{Q}_2v_r\vec{P}v_{q-1}v_n\overleftarrow{P}v_qv_p\overleftarrow{P}v_{k+1}v_1$$

is a Hamilton cycle, a contradiction.

Hence, we may assume $p+2 \leqslant j=r-1$. If there is an integer $m \in \{p+1, \ldots, j-1\}$ such that $v_m v_r \in E(G)$, then we obtain a contradiction in the same way as above. Therefore, by the choice of v_j, $|N(w)| \leqslant t-1$ for all $w \in T-\{v_p, v_{r-1}\}$, contradicting the 1-2-3-condition (recall that $t \geqslant 3$ since $p+2 \leqslant j=r-1$).

If $T=\{v_{p+1}, v_{p+2}, \ldots, v_{r-1}\}$, then $t \geqslant 1$ and $|N(w)| \geqslant t+2$ for some $w \in T$, since G satisfies the 1-2-3-condition. We then proceed in the same way as above. This time we obtain that $|N(w)| \leqslant t$ for all vertices $w \in T-\{v_{r-1}\}$, contradicting the 1-2-3-condition (recall that $t \geqslant 2$ since $p+2 \leqslant j=r-1$).

This completes the proof of Case 1.

If $|D| \geqslant 2$, suppose that $T \cap A \neq \emptyset$ and $T \cap B \neq \emptyset$. By (9) there exist $p \in \{4, \ldots, i^*\}$ and $q \in \{j^*, \ldots, n-3\}$ such that $v_{p-1}, v_{q+1} \in T$ and $v_1v_p, v_qv_n \in E(G)$. Then by (9), $v_{p-1}v_{q+1} \in E(G)$ and $v_1\vec{P}v_{p-1}v_{q+1}\vec{P}v_nv_q\overleftarrow{P}v_pv_1$ is a Hamilton cycle, a contradiction. Hence, we may assume $T \cap B=\emptyset$.

Case 2. $|D|=2$. If there is an edge v_pv_q with $p \in \{2, \ldots, i^*-1\}$ and $q \in \{j^*+1, \ldots, n-1\}$, then we proceed as in Case 1. Otherwise, since G is 2-connected, there exist integers $p \in \{2, \ldots, i^*-1\}$ and $q \in \{j^*+1, \ldots, n-1\}$ such that $v_pv_{j^*}$, $v_{i^*}v_q \in E(G)$. Note that $j^*=i^*+1$ and that $v_{q-1}v_n \in E(G)$ since $T \cap B=\emptyset$. As in Case 1, let $r=\min\{j>p \mid v_j \in N(v_1)\}$.

We now follow the proof of Case 1 (precisely). Note that $v_mv_{j^*} \notin E(G)$ for $m=p+1, \ldots, r-1$, by the minimality of $r-p$. There is a path $Q=v_pv_{j^*}\vec{P}v_{q-1}v_n\overleftarrow{P}v_qv_{i^*}$ from v_p to v_{i^*} containing v_p and all vertices of $v_{i^*}\vec{P}v_n$. Whenever we reach a contradiction in Case 1 by indicating a Hamilton cycle C of G, we can obtain a similar contradiction by replacing $v_p\vec{C}v_{i^*}$ or $v_p\overleftarrow{C}v_{i^*}$ by Q.

This completes the proof of Case 2.

Case 3. $|D| \geqslant 3$. We distinguish the two subcases $T \cap A=\emptyset$ and $T \cap A \neq \emptyset$.

I. $T \cap A=\emptyset$.

If there exist $p \in \{2, \ldots, i^*-1\}$ and $q \in \{j^*+1, \ldots, n-1\}$ such that $v_pv_q \in E(G)$, then $v_1\vec{P}v_pv_q\vec{P}v_nv_{q-1}\overleftarrow{P}v_{p+1}v_1$ is a Hamilton cycle, a contradiction. Now the 2-connectedness of G implies there exist $p \in \{2, \ldots, i^*-1\}$, $q \in \{j^*+1, \ldots, n-1\}$, $s \in \{i^*+1, \ldots, j^*\}$ and $t \in \{i^*, \ldots, j^*-1\}$ such that v_pv_s, $v_tv_q \in E(G)$. Choose s as large as possible and t as small as possible subject to the conditions, and consider two subcases.

Ia. $s \leqslant t$.

If $i^*+2 \leqslant s$ and $t \leqslant j^*-2$, then $v_{s-1}v_{t+1} \in E(G)$ by (9), and

$$v_1 \vec{P} v_p v_s \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{t+1} v_{s-1} \overleftarrow{P} v_{p+1} v_1$$

is a Hamilton cycle, a contradiction. Hence, we may assume $s=i^*+1$ and $t \leqslant j^*-1$. Since G is 2-connected, there exists an integer $i \in \{s+1, \ldots, j^*\}$ such that $v_{i^*}v_i \in E(G)$. If $i=t+1$, then $v_1 \vec{P} v_p v_s \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{t+1} v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Hence $i \neq t+1$.

Suppose $i \in \{s+1, \ldots, t\}$. If $t \leqslant j^*-2$, then, by (9), $v_{i-1}v_{t+1} \in E(G)$ and $v_1 \vec{P} v_p v_s \vec{P} v_{i-1} v_{t+1} \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Therefore, $t=j^*-1$. Since G is 2-connected, there exists an integer $j \in \{s, \ldots, t-1\}$ such that $v_j v_{j^*} \in E(G)$. If $i \leqslant j$, then $v_{i-1}v_{j+1} \in E(G)$ (by (9)), and if $i>j$, then $v_{j-1}v_{i+1} \in E(G)$. In these cases we obtain, respectively, the following Hamilton cycles contradicting the assumption:

$$v_1 \vec{P} v_p v_s \vec{P} v_{i-1} v_{j+1} \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{j^*} v_j \overleftarrow{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$$

and

$$v_1 \vec{P} v_p v_s \vec{P} v_{j-1} v_{i+1} \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{j^*} v_j \vec{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1.$$

Now suppose $i \in \{t+2, \ldots, j^*\}$. If $s<t$, then, by (9), $v_{t-1}v_{i-1} \in E(G)$ and $v_1 \vec{P} v_p v_s \vec{P} v_{t-1} v_{i-1} \overleftarrow{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Therefore, $s=t$. If $i \leqslant j^*-2$, then, by (9), $v_{t+1}v_{i+1} \in E(G)$ and

$$v_1 \vec{P} v_p v_s v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{i+1} v_{s+1} \vec{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$$

is a Hamilton cycle, a contradiction. Hence, $i \in \{j^*-1, j^*\}$. Choose the smallest possible i.

Suppose $i=j^*-1$. If there exist integers $k \in \{t+1, \ldots, i-1\}$ and $r \in \{j^*+1, \ldots, n-1\}$ such that $v_k v_r \in E(G)$, then, by (9), there is a path Q from v_t to v_k containing all vertices of $\{v_t, \ldots, v_{i-1}\}$. Then $v_1 \vec{P} v_p v_s \vec{Q} v_k v_r \vec{P} v_n v_{r-1} \overleftarrow{P} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. If there is an integer $k \in \{t+1, \ldots, i-1\}$ such that $v_k v_{j^*} \in E(G)$, then by (9), there is a path Q from v_i to v_{j^*} containing all vertices of $\{v_{t+1}, \ldots, v_{j^*}\}$. Then $v_1 \vec{P} v_p v_s v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{j^*} \overleftarrow{Q} v_i v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Hence, $N(w)-T=\emptyset$ for all vertices $w \in T-\{v_t, v_i\}$, contradicting the 1-2-3-condition.

We conclude that $i=j^*$. By the choice of i and s, and by the 1-2-3-condition, there exist integers $k \in \{t+1, \ldots, i-1\}$ and $r \in \{j^*+1, \ldots, n-1\}$ such that $v_k v_r \in E(G)$. Like in the case $i=j^*-1$ above, we can indicate a Hamilton cycle, a contradiction.

Ib. $t<s$.

If $i^*+2 \leqslant t$ and $s \leqslant j^*-2$, then $v_{t-1}v_{s+1} \in E(G)$ by (9), and

$$v_1 \vec{P} v_p v_s \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{s+1} v_{t-1} \overleftarrow{P} v_{p+1} v_1$$

is a Hamilton cycle, a contradiction. Hence, we may assume $t=i^*+1$ and $s \leqslant j^* \leqslant 1$. If $s=t+1$, then $v_1 \vec{P} v_p v_s \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction.

Hence $s \geq t+2$. We may also assume that s and t are chosen in such a way that $s-t$ is as small as possible (although this may conflict with the choice of s being as large as possible and t being as small as possible).

If there exists an integer $k \in \{t+1, \ldots, s-1\}$ such that $v_{i^*}v_k \in E(G)$, then there is a path Q from v_{i^*} to v_t containing all vertices of $\{v_{i^*}, \ldots, v_{s-1}\}$. Then $v_1 \vec{P} v_p v_s \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{Q} v_{i^*} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction; if there is such a k with $v_k v_{s+1} \in E(G)$, then there is a path from v_s to v_{s+1} containing all vertices of $\{v_{t+1}, \ldots, v_{s+1}\}$, so that $v_1 \vec{P} v_p v_s \vec{Q} v_{s+1} \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. By (9), this implies that $s = j^* - 1$. Now, however, $N(w) - T = \emptyset$ for all $w \in T - \{v_t, v_s\}$, contradicting the 1-2-3-condition.

II. $T \cap A \neq \emptyset$.

First assume there is no edge $v_p v_q$ with $p \in \{2, \ldots, i^*-1\}$ and $q \in \{j^*+1, \ldots, n-1\}$. Let $v_p \in T \cap A$ such that $v_{p+1} \notin T \cap A$. Since G is 2-connected, there are integers $q \in \{j^*+1, \ldots, n-1\}$ and $t \in \{i^*, \ldots, j^*-1\}$ such that $v_t v_q \in E(G)$. If $t \leq j^*-2$, then, by (9), $v_p v_{t+1} \in E(G)$. Then $v_1 \vec{P} v_p v_{t+1} \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Hence $t = j^*-1$. Now there exists an integer $k \in \{2, \ldots, j^*-2\}$ such that $v_k v_{j^*} \in E(G)$.

If $v_1 v_{k+1} \in E(G)$, then $v_1 \vec{P} v_k v_{j^*} \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_{k+1} v_1$ is a Hamilton cycle, a contradiction. Thus $v_{k+1} \in T$. If $v_{k+1} \in T \cap D$, then, by (9), $v_p v_{k+1} \in E(G)$ and $v_1 \vec{P} v_p v_{k+1} \vec{P} v_t v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{j^*} v_k \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction. Thus $v_{k+1} \in T \cap A$, and, by (9), $v_{k+1} v_{i^*+1} \in E(G)$. Now $v_1 \vec{P} v_k v_{j^*} \vec{P} v_{q-1} v_n \overleftarrow{P} v_q v_t \overleftarrow{P} v_{i^*+1} v_{k+1} \vec{P} v_{i^*} v_1$ is a Hamilton cycle, a contradiction.

We conclude that there exist integers $p \in \{2, \ldots, i^*-2\}$ and $q \in \{j^*+1, \ldots, n-1\}$ such that $v_p v_q \in E(G)$ and $v_{p+1} \in T \cap A$ (if $v_{p+1} \notin T$, then $v_1 v_{p+1} \in E(G)$ and $v_1 \vec{P} v_p v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{p+1} v_1$ is a Hamilton cycle, a contradiction). Then, by (9), $v_{p+1} v_{i^*+1} \in E(G)$ and $v_1 \vec{P} v_p v_q \vec{P} v_n v_{q-1} \overleftarrow{P} v_{i^*+1} v_{p+1} \vec{P} v_{i^*} v_1$ is a Hamilton cycle, our final contradiction. □

Proof of Theorem 2.1. If G is Hamiltonian, then clearly $G+uv$ is Hamiltonian. Conversely, suppose that G is not Hamiltonian, while $G+uv$ is Hamiltonian. Then the vertices of G are contained in a Hamilton path $u = v_1 v_2 \cdots v_n = v$. Let i^* and j^* be defined as before. By Lemma 3.2, $i^* > j^*$. There at least $m = \max(1, \lambda_{uv} - 1)$ constrained cycles $C_1, \ldots, C_m$ in G which induce pairwise disjoint subsets $X_1, \ldots, X_m$ of $V(G)$ with $X_i = V(G) - V(C_i)$ $(i = 1, \ldots, m)$. Among all constrained cycles we can choose one which leaves out X such that the conditions of Lemma 3.1 are satisfied. This can be seen as follows: If $\lambda_{uv} \leq 2$, then (6) is required for all vertices $w \in T$; if $\lambda_{uv} \geq 3$, then notice that, since $|X_i \cap T| \geq 1$ $(i = 1, \ldots, m)$, it suffices to require (5) for at least $t - ((\lambda_{uv} - 1) - 1) = t + 2 - \lambda_{uv}$ vertices $w \in T$. By Lemma 3.1, G is Hamiltonian, a contradiction. This completes the proof. □

Acknowledgment

The authors would like to thank Professor A. Ainouche for very helpful discussions.

References

[1] A. Ainouche and N. Christofides, Semi-independence number of a graph and the existence of hamiltonian circuits. Discrete Appl. Math. 17 (1987) 213–221.

[2] J.-C. Bermond, Hamiltonian graphs, in: L.W. Beineke and R.J. Wilson, eds., Selected Topics in Graph Theory (Academic Press, New York, 1978) 127–167.

[3] J.A. Bondy and V. Chvátal, A method in graph theory, Discrete Math. 15 (1976) 111–135.

[4] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Macmillan, London and Elsevier, New York, 1976).

[5] V. Chvátal, Hamiltonian cycles, in: E.L. Lawler, J.K. Lenstra, A.H.G. Rinnooy Kan and D.B. Shmoys, eds., The Traveling Salesman Problem (Wiley, Chichester, 1985) 403–429.

[6] R.J. Faudree, R.J. Gould, M.S. Jacobson and L. Lesniak, Neighborhood closures for graphs, Colloq. Math. Soc. Janos Bolyai 52 (1987) 227–237.

[7] E. Flandrin, H.A. Jung and H. Li, Hamiltonism, degree sum and neighborhood intersections, Discrete Math. 90 (1991) 41–52.

[8] O. Ore, Note on hamiltonian circuits, Amer. Math. Monthly 67 (1960) 55.